A NOTE ON ASSYMMETRIC GRAPHS

BY
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ABSTRACT

We describe a partition of the points of a graph which is related to its automorphism group. We then prove that the group of a tree is trivial if and only if this partition is the trivial one, and we formulate an algorithm which produces such a partition. Some application to graphs in general are also considered.

1. Introduction

In this note we consider the problem of determining when the automorphism group of a given graph, most often a tree, is trivial. In Section 2, we give a a condition which is necessary and sufficient for the group of a tree to be trivial (Theorem 2.4) and which is sufficient for the triviality of the group of an arbitrary graph. In Section 3 we describe a simple algorithm which determines for any given tree whether or not the conditions of Theorem 2.4 are satisfied. We also show that the same algorithm may be applied to an arbitrary graph and will under certain conditions determine whether or not the automorphism group is trivial.

We conclude this section with some remarks on terminology and notation. All graphs considered are finite, undirected, without loops or multiple edges. The elements of a graph are called *points* and the *order* of a graph is the number of its points. If a is adjacent to b we designate this either by the notation $a \, adj \, b$ or by saying that the (unordered) edge (a,b) is in the graph. We will say that (a,b) is an *edge at a* (or *at b*). The number of edges at a point a is called the *valency of a*, designated by $v(a)$. The set of all point adjacent to a given point a is called the *star of a*. A *path* is a sequence of points $a_1, a_2, \cdots, a_n$ with $a_i \, adj \, a_{i+1}$ and the edge (a_i, a_{i+1}) distinct from (a_j, a_{j+1}) for $i \neq j$, $1 \leqq i,j \leqq n-1$. If a and b are distinct points of the graph G, and there is at least one path

* Work supported in part by NSF Grant GP 11618.
Received January 25, 1971 and in revised form April 24, 1971

$a = a_1, a_2, \cdots, a_n = b$, then we say that the distance from a to b (or from b to a) is the minimum of the lengths of all such paths. A *cycle* is a sequence of distinct points $a_1, \cdots a_n$, $n \geqq 3$, in which $a_i \, adj \, a_{i+1}$, $1 \leqq i \leqq n-1$, and $a_1 \, adj \, a_n$. A graph is called *assymmetric* if its only automorphism is the trivial one. A graph is called a *tree* if it contains no cycles and is connected.

2. The star partition

Let G be a graph and α an automorphism on G. Then the orbits of α provide a partition of the points of G which has certain special properties.

LEMMA 2.1. *Let α be an automorphism of the graph G. Let $\mathscr{P}$ be the partition of the points of G whose elements are the orbits of α. Then $\mathscr{P}$ satisfies the following properties:*

i) *If $A \in \mathscr{P}$, then all elements of A have the same valency.*

ii) *Let $a \in A \in \mathscr{P}$ and $b \in B \in \mathscr{P}$ with $a \, adj \, b$. Then for each $a' \in A$ there is $b' \in B$ such that $a' \, adj \, b'$.*

PROOF. Since an automorphism of a graph preserves valency i) is immediate. Let a, a', b, A, B be given as in ii). Then since a and a' are in the same orbit of α there is some power of α, say α^x which maps a to a'. Thus since $a \, adj \, b$ it follows that $\alpha^x(a) \, adj \, \alpha^x(b)$. But $\alpha^x(a) = a'$ and $\alpha^x(b)$ must be in the same orbit of α as b. Hence, let $\alpha^x(b) = b' \in B$ and so $a' \, adj \, b'$.

It is worth noting that even though every partition induced by an automorphism satisfies these two properties it is not true that every such partition is the set of orbits of *some* automorphism. However, this partition is of sufficient importance to name.

DEFINITION. Let G be a graph and $\mathscr{P}$ a partition of its points satisfying the following properties:

i) If $A \in \mathscr{P}$, then all elements of A have the same valency.

ii) Let $a \in A \in \mathscr{P}$ and $b \in B \in \mathscr{P}$ with $a \, adj \, b$. Then for each $a' \in A$ there is $b' \in B$ such that $a' \, adj \, b'$.

We call such a partition a *star partition of G.*

Every graph G has a star partition, the partition whose elements consist of singleton sets. We call such a partition the *trivial partition.* We can now restate Lemma 2.1 as a condition on the triviality of the automorphism group of a graph.

THEOREM 2.2. *Let G be a graph whose only star partition is the trivial partition. Then G is assymmetric.*

The converse of this theorem is not true since there exist graphs each of whose points have the same valency but which are assymmetric. In such a case a non-trivial star partition consists of one element containing all of the points of G. In the case of trees, however, the converse is true.

THEOREM 2.3. *Let G be a tree with a non-trivial star partition. Then the automorphism group of G is not trivial.*

We can now put Theorems 2.2 and 2.3 together to get

THEOREM 2.4. *A tree is asymmetric if and only if it has only the trivial star partition.*

Before giving the proof of Theorem 2.3 we need to introduce some terminology and prove some preliminary lemmas.

DEFINITION. Let G be a graph (without loops or multiple edges) and $\mathscr{P}$ a star partition of G. Then we may regard $\mathscr{P}$ as a graph with A *adj* B, $A \neq B$ if there is $a \in A$ and $b \in B$ such that a *adj* b. We call $\mathscr{P}$ a *star partition graph of* G, and denote it by $\bar{\mathscr{P}}$.

It follows directly from the definition of a star partition that the star partition graph is in fact a graph. (The condition that $A \neq B$ guarantees that $\bar{\mathscr{P}}$ has no loops.)

LEMMA 2.5. *A star partition graph of a tree is a tree.*

PROOF. Let G be a tree and $\mathscr{P}$ a star partition of G. Suppose that the star partition graph $\bar{\mathscr{P}}$ contains the cycle $A_1, \cdots, A_n$. Then there is a sequence of points $a_1, a_2, \cdots, a_n, a_1'$ with $a_i \, adj \, a_{i+1}$, $1 \leqq i \leqq n-1$, $a_n \, adj \, a_1'$ and $a_i \in A_i$, $a_1' \in A_1$. If $a_1' = a_1$ then we have a cycle in G contradicting the fact that G is a tree. If $a_1' \neq a_1$, then we can find a sequence $a_1', a_2', \cdots, a_n', a_1''$ similarly. If any of these is the same as in the previous sequence, again we have produced a cycle in G, a contradiction. If not, we simply continue to iterate this process and since G is finite we will eventually produce a cycle in G. Thus, the star partition graph contains no cycles. Finally since G is connected, $\bar{\mathscr{P}}$ is connected and so is a tree.

Since the partition $\mathscr{P}$ of G will be our main focus of attention it is necessary to introduce a refinement of our notion of valency relative to $\mathscr{P}$.

DEFINITION. Let $\mathscr{P}$ be a star partition of the graph G. Let $A, B \in \mathscr{P}$ and let

$a \in A$. (A need not be different from B). The *valency of a relative to B* is the number of points of B which are adjacent to a. We denote this integer by $v_B(a)$.

The next two lemmas are technical ones which will be needed in the proof of 2.3.

LEMMA 2.6. *Let $\mathscr{P}$ be a star partition of the tree G and let $a \in A \in \mathscr{P}$ with $v_A(a) > 1$. Then there is $B \in \mathscr{P}$, $B \neq A$, $c \in A$, and $e, f \in B$ such that*

i) *There is a path in A from a to c,*

ii) *$v_A(c) = 1$ and*

iii) *$c \operatorname{adj} e$, $c \operatorname{adj} f$.*

PROOF. The set of points in A span a subgraph of G which must therefore be a graph with no cycles. Hence, in the connected component of this subgraph which contains a there must be a point of valency 1 since the valency of a in this subgraph is greater than 1. Call such a point c. Now the valency of any point, in particular $a \in A$, is given by $v(a) = \sum_{A_i \in \mathscr{P}} v_{A_i}(a)$. This sum can, of course, be restricted to those A_i which are adjacent to A in $\mathscr{P}$ plus A itself. But this set of A_i's is the same for c as it is for a and hence if $v_{A_i}(c) = 1$ for all A_i's adjacent to A is would follow that the valency $v(a) > v(c)$. This contradicts the fact that $v(a) = v(c)$ since a and c are in the same element of a star partition. Thus there must be some $B \in \mathscr{P}$ such that $v_B(c) > 1$ (and hence $B \neq A$).

LEMMA 2.7. *Let $\mathscr{P}$ be a star partition of the graph G. Let $a, a' \in A \in \mathscr{P}$ with $v_A(a') = 0$ and $v_B(a) \geqq 2$ for some $B \in \mathscr{P}$. Then there is $B' \in \mathscr{P}$ such that $v_{B'}(a') \geqq 2$.*

PROOF. Since $a, a' \in A$ it follows that $v(a) = v(a')$. If $v_C(a') < 2$ for all $C \in \mathscr{P}$, then since $v_C(a) \cdot v_C(a') = 0$ if and only if $v_C(a) = v_C(a') = 0$, there must be $B' \in \mathscr{P}$ such that $v_{B'}(a') \geqq 2$.

Finally, we will need to deal with a special sort of subtree in producing an automorphism of the tree G.

DEFINITION. Let a be a point of a tree G with $v(a) \geqq 2$. We produce a subgraph of G in the following manner. Delete all the edges incident at a except 2 of them. Now consider the connected component of the reduced graph which contains the point a. We call this subgraph a *bi-tree at a*.

LEMMA 2.8. *Let G be a tree, a a point of G and H a bi-tree at a of G. Then any automorphism of H which fixes a can be extended to an automorphism of G by defining it to fix all points of G not in H.*

PROOF. Let α' be an automorphism of H with $\alpha'(a) = a$. Now define α on G by $\alpha(x) = \alpha'(x)$ when $x \in H$ and $\alpha(x) = x$ otherwise. To verify that α is an automorphism of G it suffices to show that if $u \, adj \, v$ (in G), then $\alpha(u) \, adj \, \alpha(v)$ (in G) and that α is one-to-one. Clearly α is one-to-one on G since it is one-to-one on H. Now if $u, v \notin H$, $u \, adj \, v$, then $\alpha(u) = u$, $\alpha(v) = v$ and so $\alpha(u) \, adj \, \alpha(v)$. If $u, v \in H$ and $u \, adj \, v$ not only in G but also in H, then clearly $\alpha(u) \, adj \, \alpha(v)$. If $u, v \in H$, $u \, adj \, v$ in G but not in H, then it follows that there are two distinct paths from u to v in G, contradicting the fact that G is a tree. Finally, if $u \in H$, $v \notin H$ and $u \, adj \, v$ in G it follows that $u = a$ and so since $\alpha(a) = a$, $\alpha(u) \, adj \, \alpha(v)$.

We are now ready to give the proof of Theorem 2.3.

PROOF OF 2.3. Let $\mathscr{P}$ be a non-trivial star partition of the tree G.

Case I. For all $a \in G$ and for all $B \in \mathscr{P}$, $v_B(a) \leqq 1$ except possibly if $a \in B$. If $A \in \mathscr{P}$, then for all $x, y \in A$, $v_A(x) = v_A(y)$. For the condition above implies that $v_B(x) = 0$ or 1 and if $v_B(x) = 1$, then $v_B(y) = 1$. Hence, since $v(x) = v(y)$ it follows that $v_A(x) = v_A(y)$. Furthermore, since G is a tree either $v_A(x) = 0$ for all $x \in A$ or $v_A(x) = 1$ for all $x \in A$. Now if each element of $\mathscr{P}$ consisted only of isolated points, then it follows that the graph G is not a tree. For let $a, b \in A \in \mathscr{P}$. Then since G is connected there is a path in G from a to b with no repetitions. Thus there is a path in the star partition graph $\bar{\mathscr{P}}$ beginning at A. But since G is a tree, $\bar{\mathscr{P}}$ is a tree and so this path beginning at A and ending at A must have a repetition in it. Let B be the first such repetition. (Such a path is, of course, unique since the path from a to b is unique.) It is easy to see that this repetition is either of the form B, B or B, C, B. In the case of B, B we must have a pair of points in B which are adjacent—a possibility ruled out by assumption. If the repetition takes the form B, C, B, then it follows that C contains a point c with $v_B(c) \geqq 2$—contrary to our initial assumption. Hence the assumption that each element of $\mathscr{P}$ consisted only of isolated points when considered as a subgraph is false. Therefor, there is an $A \in \mathscr{P}$ with $v_A(x) = 1$ for all $x \in A$. Now we will show that there is precisely one such element in $\mathscr{P}$. For let $A, B \in \mathscr{P}$ with $v_A(x) = 1$ for all $x \in A$ and $v_B(y) = 1$ for all $y \in B$. Now there is a unique path in the graph $\bar{\mathscr{P}}$ from A to B say $A = A_1, A_2, \cdots, A_r = B$ and hence for each point in A there is a path which contains precisely one point from each of A_i, $i = 1, \cdots, r$. Furthermore, we can guarantee that these paths are all disjoint by choosing A and B in such a way that for each A_i, $1 < i < n$, $v_{A_i}(x) = 0$ for all $x \in A_i$. Thus, this set of paths (points and edges) together with the edges "inside" A and "inside" B constitutes a subgraph of G in which each point has valency 2—contradicting that G is a tree. Now let A be the unique element of $\mathscr{P}$ with $v_A(x) = 1$ for all $x \in A$. We now show that A consists precisely of two points a, a' with $a \, adj \, a'$. Clearly, there are such a pair of points in A. Assume there is another point $a_2 \in A$. Since G is connected, there is a path from a_2 to a in G and hence a path from A to A in $\bar{\mathscr{P}}$ consisting of more than A itself. But since every point in B, $B \neq A$ has valency 0 in B and $v_C(x) \leqq 1$ for all $C \in \mathscr{P}$ it follows that such a path from A to A must be a cycle contradicting the fact that G is a tree. Hence, A consists only of a, a' with $a \, adj \, a'$. Further, every other $B \in \mathscr{P}$ consists of a pair of points of valency 0 in B. For if $B \in \mathscr{P}$, then there is a path from B to A in $\mathscr{P}$ and hence a point b in B connected to, say a by a path not containing a'. Clearly there is therefore a point $b' \in B$, $b' \neq b$, such that b' is connected to a' by a path not containing a. Now if there were a third point $b'' \in B$, then G would have to contain a point with relative valence greater than one, contradicting our assumption. Now we define the map $\alpha: G \to G$ by $\alpha(a) = a'$ and $\alpha(b) = b'$ where $b, b' \in B \in \mathscr{P}$, b and b' are respectively the points connected to a and a' as above. The verification that α is an automorphism of G is direct and is left for the reader.

Case II. There is a point $a \in G$ and $B \in \mathscr{P}$ such that $v_B(a) > 1$ and $a \notin B$. Thus, there are points $b, c \in B$ with $a \, adj \, b$ and $a \, adj \, c$. Hence, we form the bi-tree at a obtained by deleting all edges at a except (a, b) and (a, c) and call it H. We will assume that a and H have been chosen in such a way that the number of points of H is minimal. We will now show that if $d \in H$, $d \neq a$, then $v_C(d) \leqq 1$ for all $C \in \mathscr{P}$. (Note, of course, that $v_C(d)$ is computed in G.) Hence, suppose that H contains a point d with $v_C(d) > 1$ for some $C \in \mathscr{P}$. First, suppose that $d \notin C$ and let $d \in D \in \mathscr{P}$. Let $e, f \in C$ with $d \, adj \, e$ and $d \, adj \, f$. We now form the bi-tree K at d by deleting all edges at d except (d, e) and (d, f). If K does not contain the point a, then it is a proper subgraph of H containing a point "like" a, thus contradicting the minimality of H. For if $k \in K$, then there is a path from k to d in K and a path from d to a in H. But if $k \notin H$, then the path from k to d must contain one of the edges deleted in the formation of H. But this implies that a is an element of the path from k to d in K and hence that $a \in K$, a contradiction. Thus, we assume that the subgraph K contains a. Let $A, D \in \mathscr{P}$ such that $a \in A$ and $d \in D$. Then there is a path from D to A in the graph $\bar{\mathscr{P}}$ and two distinct paths from e and f consisting of one point each from this path in $\bar{\mathscr{P}}$ from e or f to a. Assume that there is such a path from e to a. Then there is a similar but disjoint path from f to some $a' \in A$, $a' \neq a$. (Moreover, the path from e must include either b or c, for if not there are two distinct paths from d to a, a contradiction.) Now, among all such points d assume that we have chosen one whose distance from a is maximum. If $v_A(a') = 0$, then since $v_B(a) \geqq 2$ it follows from Lemma 2.7 that there is $B' \in \bar{\mathscr{P}}$ such that $v_{B'}(a') \geqq 2$. Now consider a bi-tree L at a' with two edges from a' to B'. If L contains a we have a contradiction to the maximality of the path from d to a since the distance from a' to a is greater than that from d to a. Hence, L does not contain a. For otherwise the path from d to a contains a_1. But this implies that L is a proper subgraph of H, a contradiction. Hence, $v_A(a') > 0$ and so a' is connected to another point $a'' \in A$, with $v_A(a'') = 1$, by a path in A. (Notice that $a'' \neq a$ since G is a tree.) Hence, there is a path through the same elements of $\bar{\mathscr{P}}$ "parallel" to the path from d to a, without duplicating any points previously used, ending with $d' \in D$. Again it follows that d' is connected in D to a point $d'' \in D$, $d'' \neq d'$, $d'' \neq d$ and there is a path from d'' to a point in A without duplicating any points previously used. Since G is finite this process must end yielding a contradiction. Hence, it follows that either $v_D(d) > 1$ or that $v_C(d) \leqq 1$ for all $C \in \mathscr{P}$. Thus, assume that $v_D(d) > 1$. It follows from Lemma 2.6 that there is $\bar{d} \in D$ connected to d by a path in D with $\bar{d}$ a point "like" a and we arrive at a contradiction by the previous argument.

Thus, we have proven that every point u, $u \neq a$, of the bi-tree H has the property $v_C(u) \leqq 1$ for all $C \in \mathscr{P}$. Now every point of H other than a is connected either to b or to c but not to both. We will designate such points as b-points or c-points, respectively. If $E \in \mathscr{P}$ and E contains points of H we claim that E contains precisely one b-point and one c-point. To see this suppose that there is a point $d \in H$, $d \neq a$, with $d \in D$ and $v_D(d) = 1$. (We have already dealt with the case $v_D(d) > 1$.) Since $d \in H$ there is a path from d to a through a unique set of elements of $\bar{\mathscr{P}}$. But since $v_D(d) = 1$ there is $d' \in D$, $d \, adj \, d'$. Thus there will be a path from d' to an element $a' \in A$ containing exactly one point in each element of the previous path in $\bar{\mathscr{P}}$. Clearly, $a' \neq a$ and $v_B(a') \leqq 1$ for all $B \in \mathscr{P}$. Thus $v_A(a') = 1$ and so there is $a'' \in A$, $a'' \neq a$, with $a' \, adj \, a''$. Hence, there is a path back to D from a'' "parallel" to the one from d' to a'. But since G is finite this process must eventually end in a point of valency 1 in G, a contradiction. Therefore, we have proven that $v_D(d) = 0$ for all $d \in H$, $d \in D$ except possible $d = a$. Thus it follows that if $A \neq D \in \mathscr{P}$ has points of H then it has exactly one b-point and one c-point. Also $H \cap A$ contains only a. We now define the map $\alpha: H \to H$ by: $\alpha(a) = a$ and if $x, y \in D$ are the b-point and c-point of D respectively, then $\alpha(x) = y$ and $\alpha(y) = x$.

Clearly, this map is an automorphism on H since the only adjacency relations are those between adjacent elements of $\mathscr{P}$ and the definition of star partition guarantees that these are preserved. We now invoke Lemma 2.8 and α is a non trivial automorphism (of order 2) on G.

REMARK 1. Although Theorem 2.3 is stated for trees it is easy to modify the proof above and prove the same result without the requirement of connectitivity. Thus the theorem holds for "forests" as well.

3. **The star algorithm**

In this section we will describe a simple algorithm which produces a star partition for any graph G. Further we will show that the orbit partition of the automorphism group of the graph is a refinement of this partition.

$\mathscr{P}_0$. The partition $\mathscr{P}_0$ is the partition of the points of G into sets of equal valency.

$\mathscr{P}_i$. Form the star of each element of $\mathscr{P}_{i-1}$. (The star of a set of points is the union of the stars of its members.) Partition these stars into sets of equal valency and add to them the elements of $\mathscr{P}_{i-1}$. This collection of sets will form a partially ordered set under inclusion. Close this partially ordered set by the operation of intersection and relative complementation. That is if A and B are elements of the partially ordered set form $A \cap B$, $A-(A \cap B)$ and $B-(A \cap B)$. The minimal non-empty elements of this closed partially ordered set will be the elements of $\mathscr{P}_i$.

$\mathscr{P}_n = \mathscr{P}$. If n is the smallest positive integer with $\mathscr{P}_{n-1} = \mathscr{P}_n$, then let $\mathscr{P}_n = \mathscr{P}$ and stop.

We will refer to the partition $\mathscr{P}$ as the *standard star partition.*

Since at each step in the algorithm above we replace a given partition by a refinement and the original set is finite it is clear that the process will stop after a finite number of steps. Clearly the elements of $\mathscr{P}_i$ consist of points of equal valency. Now suppose that $A, B \in \mathscr{P}$ with $a \in A$, $b \in B$ and $a \, adj \, b$. Let $a' \in A$ and suppose that there is no element $b' \in B$ such that $a' \, adj \, b'$. Then the star of B, B^*, will not contain the element a' and hence $A \cap B^*$ will be a proper subset of A. This implies that at some stage in the algorithm A will be replaced by proper subsets and hence $A \notin \mathscr{P}$. Hence we have proven the following theorem.

THEOREM 3.1. *Let G be a graph and $\mathscr{P}$ the standard star partition of G. Then $\mathscr{P}$ is a star partition of G.*

Furthermore, we will show that the algorithm provides the coarsest of all star partitions.

THEOREM 3.2. *Let $\mathscr{P}$ be the standard star partition of the graph G. Let $\mathscr{P}'$ be another star partition of G. Then for each $A' \in \mathscr{P}'$ there is $A \in \mathscr{P}$ such that $A' \subseteq A$.*

PROOF. We will prove the theorem by induction on the number of steps needed for the completion of the algorithm. First, since the elements of $\mathscr{P}'$ consist of points of equal valency the theorem is true for $\mathscr{P}_0$. Now assume it is true for $\mathscr{P}_i$ and let $A' \in \mathscr{P}'$, $A \in \mathscr{P}_i$ with $A' \subseteq A$. Let $B \in \mathscr{P}_i$ and consider $A \cap (star\ of\ B)$. We must show that either $A' \subseteq A \cap (star\ of\ B)$ or that $A' \cap A \cap (star\ of\ B) = \emptyset$. Let $a' \in A'$ with $a' \in (star\ of\ B)$ and let a'_1 be another point in A'. Since $a' \in (star\ of\ B)$, there is a $b' \in B$ with $a'\ adj\ b'$ and $B' \in \mathscr{P}'$ with $b' \in B'$. By the induction assumption $B' \subseteq B$ and so there is a $b'_1 \in B' \subseteq B$ with $a'_1\ adj\ b'_1$. Hence $A' \subseteq (star\ of\ B)$. This completes the proof.

COROLLARY 3.3. *Let G be a graph If the standard star participation of G is trivial (consists only of singleton sets), then G is assymmetric.*

PROOF. Since the partition of G into the orbits of its automorphism group is a star partition, it follows from Theorem 3.2 that it will be a refinement of its standard star partition. Thus, if the standard star partition is trivial the orbits of the automorphism group are singleton sets and the graph G is assymmetric.

REMARK 2. The main result of the previous section shows that when the automorphism group of the tree G is trivial, then a star partition must be trivial and hence gives the orbits of the automorphism group. It is in general not the case, however, that the elements of the standard star partition are the orbits of the automorphism group. For let G be the graph

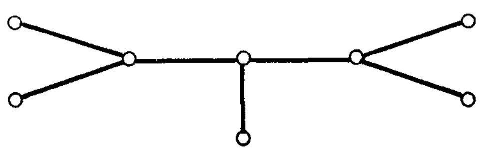


Fig. 1

Then the following is the standard star partition

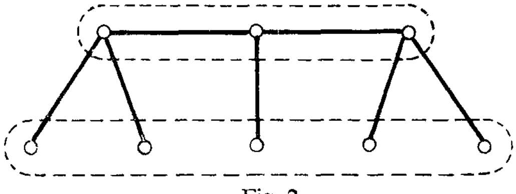


Fig. 2

But it is easy to see that the partition of G into the orbits of its automorphism group is given by

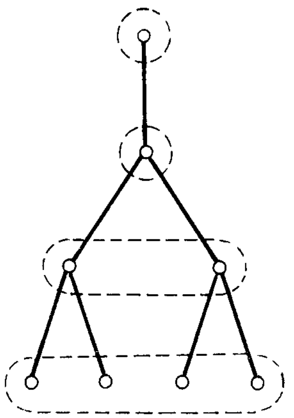
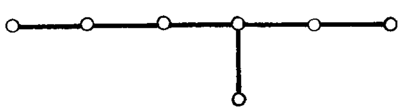


Fig. 3

REMARK 3. The star algorithm given above was formulated for maximum clarity and simplicity and not necessarily for speedy computability. It is not difficult, however, to refine the algorithm to yield a more efficient procedure for arriving at the standard star partition.

REMARK 4. The author first used the star algorithm in testing a conjecture about the automorphism group of a tensor product of a pair of assymmetric graphs. (See [1] for a definition of tensor product). The graphs in question were



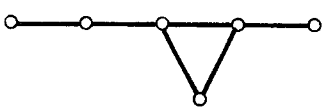


Fig. 4

The tensor product is not a tree or even a forest. Applying the star algorithm does, however, yield a trivial partition and so shows it to be assymmetric. This raises the question about which graphs are assymmetric but have a non-trivial star partition.

We hope to pursue these questions in a later note.

4. Acknowledgement

We acknowledge with great pleasure some very helpful conversations with Dr. Benjamin Weiss.

REFERENCE

1. F. Harary and G. Wilcox, *Boolean operations on graphs*, Math. Scand. **20** (1967), 41–51.

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